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# GENERALIZED SINGULAR FUNCTIONS

BY

G. KÄLLÉN AND H. WILHELMSSON

København 1959
i kommission hos Ejnar Munksgaard

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# CONTENTS



## Synopsis


It is shown that the generalized singular function

$$\Delta_{n+1}^{(+)}(x;a) = \frac{(-i)^n}{(2\pi)^{3n}} \int \cdots \int dp_1 \ldots dp_n \, e^{i\,\Sigma\, p_k x_k}\, \Pi\, \delta(p_k p_l + a_{kl})\, \Pi\, \theta(p_k)$$

for arbitrary values of $n$ can be expressed in terms of the special function $\Delta_5^{(+)}(x;a)$. For $n > 4$, this follows from the fact that ordinary space time has four dimensions and, therefore, more than four vectors are always linearly dependent. The basic function $\Delta_5^{(+)}(x;a)$ is treated in some detail. It is shown that the sixteen integrations in the definition of this function can be reduced to one integration. The representation of the function $\Delta_5^{(+)}(x;a)$ obtained in this way has a kernel consisting of a Hankel function multiplied by elementary functions only. This representation can, in principle, be used to determine the analyticity domains for all the functions $\Delta_{n+1}^{(+)}(x;a)$.



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## Introduction

The characterization of a function $f(x)$ with a Fourier transform $f(k) = \int dx\, e^{-ikx} f(x)$ that vanishes, except for positive values of $k$, is a well-known mathematical problem with many applications in theoretical physics. The solution of this problem is a function $f(x)$ which is the boundary value of an analytic function of $x$, regular in the half plane $\mathrm{Im}\,(x) > 0$. In the theory of quantized fields one is interested in a relativistic generalization of this problem, which can be formulated in the following way: What are the properties of an invariant function $F(x)$ depending on a four-vector $x = (x_1, x_2, x_3, x_4 = ix_0)$ and with a Fourier transform $F(k)$ that vanishes except when the four-vector $k$ lies in the forward light cone, i. e., when $k^2 = k_1^2 + k_2^2 + k_3^3 - k_0^2 = \overline{k}^2 - k_0^2 < 0$ and $k_0 > 0$? The case where the function $F(x)$ does not depend on any other four-vector except $x$ is comparatively simple, but considerable complications arise when the function $F(x)$ depends also on other four-vectors.

Questions of this kind are of special interest in connection with the properties of vacuum expectation values of products of field operators in different space-time points. The mathematical structure of vacuum expectation values of this kind has been the subject of some recent investigations[1,2,3]. In particular it was shown in ref. 2 that the vacuum expectation value of the product of $n$ (scalar) fields $A_1\,(x_1)\ldots A_n\,(x_n)$ is the boundary value of an analytic function depending only on the Lorentz invariant variables $z_{ik} = -(x_i - x_{i+1})\,(x_k - x_{k+1})$. This analytic function is regular in a certain domain $\mathfrak{M}$ that is obtained if one adds an imaginary vector $\eta_i$ to the coordinate difference $x_i - x_{i+1}$ and lets the $\eta_i$'s vary independently inside the forward light cone. This result is a consequence of the following two simple assumptions:

I. The theory is invariant under Lorentz transformations.

II. The energy-momentum spectrum of the theory contains vectors only in the forward light cone[4].

[1] Cf. e. g. H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. **6**, 543 (1951); G. Källén, Helv. Phys. Acta **25**, 417 (1952); H. Lehmann, Nuovo Cimento **11**, 342 (1954); M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[2] A. Wightman, Phys. Rev. **101**, 860 (1955); D. Hall and A. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. **31**, no. 5 (1957).

[3] G. Källén and A. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, no. 6 (1958). The last paper is referred to as KW below.

[4] For further details, see e. g. the introduction of KW.

If one imposes the further condition:

III. A field operator at a point $x$ commutes with a field operator at a point $x'$ if the distance between $x$ and $x'$ is space-like, i. e. if $(x-x')^2>0$,

the domain of analyticity of the analytic functions belonging to the vacuum expectation values is, in general, enlarged.

The special case of three field operators was considered in some detail in KW. The regularity domain $\mathfrak{M}$ for that case was explicitly computed and shown to be bounded by so-called "analytic hypersurfaces"[3]. The actual calculation of $\mathfrak{M}$ was made in three independent ways[5], one of which consisted in getting an integral representation of the most general function fulfilling the postulates I and II above and investigating the analytic properties of that representation. The purpose of the present paper is to generalize this representation of the product of three field operators to a representation of a product of $n$ operators. Our hope is that such a representation of the $n$-fold expectation value will be as useful for the explicit determination of the analyticity domain for the corresponding analytic function as turned out to be the case with the three-fold vacuum expectation value. However, the actual applications along these lines are not dealt with in this paper. We note that the vacuum expectation value of $n+1$ (scalar) field operators can be written as a sum over "intermediate states" $|z\rangle$ in the following way (cf. KW):

$$\langle 0 \,|\, A_1(x_1)\ldots A_{n+1}(x_{n+1}) \,|\, 0\rangle = \sum_{|z_1\rangle\ldots|z_n\rangle} e^{i\Sigma p^{(z_k)}\xi_k} \langle 0 \,|\, A_1 \,|\, z_1\rangle \langle z_1 \,|\, A_2 \,|\, z_2\rangle \ldots \langle z_n \,|\, A_{n+1} \,|\, 0\rangle . \quad (1)$$

Here, $p^{(z_k)}$ is the energy-momentum vector of the state $|z_k\rangle$,

$$\xi = x_k - x_{k+1} \quad \text{and} \quad \langle z_{k-1} \,|\, A_k(x_k) \,|\, z_k\rangle = \langle z_{k-1} \,|\, A_k \,|\, z_k\rangle \, e^{i\,(p^{(z_k)}-p^{(z_{k-1})})\,x_k}.$$

Equation (1) means that, if we introduce the Fourier transform $G^{A_1\ldots A_n}(p_1\ldots p_n)$ of the vacuum expectation value $\langle 0 \,|\, A_1(x_1)\ldots A_{n+1}(x_{n+1}) \,|\, 0\rangle$ in the following way,

$$\langle 0 \,|\, A_1(x_1)\ldots A_{n+1}(x_{n+1}) \,|\, 0\rangle = \frac{1}{(2\pi)^{3n}} \int\ldots\int dp_1\ldots dp_n \, e^{i\Sigma p_k \xi_k} G^{A_1\ldots A_n}(p_1\ldots p_n), \quad (2)$$

the function $G^{A_1\ldots A_n}(p_1\ldots p_n)$ is different from zero only when *all* vectors $p_k$ lie in the forward light cone. Because of Lorentz invariance we can therefore write

$$G^{A_1\ldots A_n}(p_1\ldots p_n) = G(p_k p_l) \prod_{k=1}^{n} \theta(p_k); \quad \theta(p_k) = \frac{1}{2}\left[1+\frac{p_{k_0}}{|p_{k_0}|}\right], \quad (3)$$

where the function $G(p_k\, p_l)$ depends only on the scalar products $p_k\, p_l = \bar{p}_k\, \bar{p}_l - p_{k_0}\, p_{l_0}$ and is different from zero only if all $p_k^2<0$. Hence, we can write

[5] Cf. section IV and appendices I and II of KW.

$$\langle 0 \mid A_1(x_1) \ldots A_{n+1}(x_{n+1}) \mid 0 \rangle = i^n \int_0^\infty \ldots \int_0^\infty \Pi\, da_{kl}\, G(-a_{kl})\, \Delta_{n+1}^{(+)}(\xi_k; a_{kl}), \tag{4}$$

with $$\Delta_{n+1}^{(+)}(x_k; a_{kl}) = \frac{(-i)^n}{(2\pi)^{3n}} \int \ldots \int dp_1 \ldots dp_n\, e^{i\Sigma p_k x_k} \prod_{k \le l} \delta(p_k p_l + a_{kl})\, \Pi\, \theta(p_k). \tag{4a}$$

Equation (4) explicitly exhibits our vacuum expectation value as a convolution integral over a "weight function" $G(-a_{kl})$ with a "generalized singular function" $\Delta_{n+1}^{(+)}(x_k; a_{kl})$[6]. The two special cases $n = 1$ and $n = 2$ are well known[7]

$$\Delta_2^{(+)}(x; a) = -\frac{a}{8\pi} H_1^{(1)}(\sqrt{az})/\sqrt{az}, \quad z = -x^2, \tag{5a}$$

$$\left.\begin{aligned} \Delta_3^{(+)}(x_1, x_2; a_{11}, a_{22}, a_{12}) = -\frac{2i}{(4\pi)^3} \frac{\sqrt{a_{12}^2 - a_{11}a_{22}}}{\sqrt{R}} \Big[ H_0^{(1)}\left(\sqrt{Q+\sqrt{R}}\right) \\ - H_0^{(1)}\left(\sqrt{Q-\sqrt{R}}\right)\Big]\, \theta\left(a_{12} - \sqrt{a_{11}a_{22}}\right), \end{aligned}\right\} \tag{5b}$$

$$Q = a_{11}z_{11} + 2a_{12}z_{12} + a_{22}z_{22}, \quad z_{kl} = -x_k x_l, \tag{5c}$$

$$R = 4\left[z_{12}^2 - z_{11}z_{22}\right]\left[a_{12}^2 - a_{11}a_{22}\right]. \tag{5d}$$

Eq. (5a) demonstrates the fact that the function $\Delta_2^{(+)}(x; a)$ is the boundary value of an analytic function regular for all values of $z$ not on the positive real axes. In a similar way, Eq. (5b) shows that $\Delta_3^{(+)}(x_k; a_{kl})$ is the boundary value of an analytic function of $z_{kl}$ regular and exponentially decreasing for large values of $|z_{kl}|$ for all points not fulfilling the relation

$$Q \pm \sqrt{R} = \varrho; \quad 0 < \varrho < \infty. \tag{6}$$

It was an exploration of Eq. (6) which led to an explicit determination of the boundary of the domain $\mathfrak{M}$ in KW. We are particularly interested in obtaining the generalization of (6) for the case $n > 2$. It should perhaps be mentioned that a representation of the kind we have in mind generates the most general function fulfilling assumptions I and II above, but that the restrictions imposed by assumption III are not taken care of. However, it is a separate problem to investigate the enlargement of the analyticity domain that follows from assumption III. (For an illustration of this statement we refer to sections V, VII, and VIII of KW.)

The two functions (5) can be calculated by straightforward methods, but a frontal attack on the higher functions leads to considerable formal complication. Fortunately, there exist relations connecting functions with different values of $n$ as

[6] Functions of this kind were first introduced by A. Wightman and D. Hall, Phys. Rev. **99**, 674 (1955).
[7] Cf. e. g. appendix I of KW.

well as invariance properties of each function $\Delta_{n+1}^{(+)}$ for a given value of $n$. The exploration of these things will allow us to simplify our calculations to a certain extent. The first three sections are devoted to a discussion of these features.

## I. Reduction of the Function $\Delta_{n+1}^{(+)}$ to $\Delta_5^{(+)}$ when $n \geq 5$

As ordinary space time does not have more than four dimensions, the vectors $p_k$ appearing in the definition (4a) cannot be linearly independent when $n \geq 5$. This allows us to perform some of the integrations in (4a) in a comparatively simple way. If we suppose that the matrix $\| a_{kl} \|$ has rank four, we can choose a set of four linearly independent vectors among the $p_k$'s. It is then only a question of labelling to call these vectors $p_1 \ldots p_4$. With the aid of the $\delta$-functions we can then express the other vectors $p_5 \ldots p_n$ as linear combinations of $p_1 \ldots p_4$ in the following way:

$$p_k = \sum_{\lambda=1}^{4} \alpha_{k\lambda} p_\lambda \quad \text{for} \quad k \geq 5. \tag{7}$$

The coefficients $\alpha_{k\lambda}$ are determined from the conditions

$$p_k p_\lambda = -a_{k\lambda} \quad \text{for} \quad k \geq 5, \quad \lambda \leq 4. \tag{8}$$

This yields

$$\alpha_{k\lambda} = \sum_{\lambda'=1}^{4} \frac{a_{k\lambda'} \Delta_{\lambda'\lambda}}{D}, \tag{9}$$

with

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix} < 0 \tag{9a}$$

and with $\Delta_{\lambda\lambda'}$ being the cofactor of $a_{\lambda\lambda'}$ in $D$. Note that the indices $\lambda$ and $\lambda'$ only take values from 1 to 4 in this definition.

After these preliminaries we can now evaluate integrals of the type

$$I = \int dp_k \prod_{l=1}^{4} \delta(p_k p_l + a_{kl}) F(p_k), \quad (k \geq 5), \tag{10}$$

where $F(p_k)$ is an arbitrary function of the vector $p_k$. We get

$$I = \frac{1}{\sqrt{-D}} F\left(\frac{1}{D} \sum_{\lambda,\lambda'=1}^{4} a_{k\lambda} \Delta_{\lambda\lambda'} p_{\lambda'}\right). \tag{11}$$

By repeated application of Eqs. (10) and (11) we find

$$\left.\begin{aligned} J &= \int\cdots\int dp_5\ldots dp_n\, e^{i\sum\limits_{k=5}^{n} p_k x_k} \mathop{\Pi}_{\substack{\lambda=1\ldots 4\\ k=5\ldots n}} \delta\left(p_k p_\lambda + a_{k\lambda}\right) \mathop{\Pi}_{k_1\le k_2=5}^{n} \delta\left(p_{k_1} p_{k_2} + a_{k_1 k_2}\right)\\ &= \frac{1}{\left(\sqrt{-D}\right)^{n-4}} \mathop{\Pi}_{k_1\le k_2=5}^{n} \delta\left(a_{k_1 k_2} - \frac{1}{D^2}\sum_{\lambda_1,\lambda_2=1}^{4}\sum_{\lambda_3,\lambda_4=1}^{4} a_{k_1\lambda_1}\Delta_{\lambda_1\lambda_2} a_{\lambda_2\lambda_3}\Delta_{\lambda_3\lambda_4} a_{\lambda_4 k_2}\right)\\ &\quad\times e^{\frac{i}{D}\sum\limits_{k=5}^{n}\sum\limits_{\lambda,\lambda'=1}^{4} a_{k\lambda}\Delta_{\lambda\lambda'} p_{\lambda'} x_k}. \end{aligned}\right\} \tag{12}$$

The arguments of the $\delta$-functions can be somewhat simplified with the aid of the relation

$$\sum_{\lambda'=1}^{4} a_{\varkappa\lambda'}\Delta_{\lambda'\lambda} = D\,\delta_{\varkappa\lambda}, \tag{13}$$

if both $\varkappa$ and $\lambda$ are $\le 4$. In this way we get

$$J = (-D)^{\frac{(n-4)^2}{2}} e^{\frac{i}{D}\sum\limits_{k=5}^{n}\sum\limits_{\lambda,\lambda'=1}^{4} a_{k\lambda}\Delta_{\lambda\lambda'} p_{\lambda'} x_k} \mathop{\Pi}_{k\le k'=5}^{n} \delta\left(a_{kk'}D - \sum_{\lambda,\lambda'=1}^{4} a_{k\lambda}\Delta_{\lambda\lambda'} a_{\lambda' k'}\right). \tag{14}$$

We now return to the integral (4a) and first remark that we can replace the product $\Pi\theta(p_k)$ by $\theta(p_1)\mathop{\Pi}\limits_{k=2}^{n}\theta(a_{1k})$. This follows from the simple observation that the scalar product of two timelike vectors is negative if they both lie in the same light cone and positive if they lie in opposite cones. Using this and (14), we can write the integral (4a) in the following way:

$$\left.\begin{aligned} \Delta_{n+1}^{(+)}(x;a) &= \mathop{\Pi}_{k=2}^{n}\theta(a_{1k})\frac{(-i)^n}{(2\pi)^{3n}}\int\cdots\int dp_1\ldots dp_4\, e^{i\sum\limits_{\varkappa=1}^{4} x_\varkappa p_\varkappa} \mathop{\Pi}_{\varkappa\le\varkappa'=1}^{4}\delta\left(p_\varkappa p_{\varkappa'} + a_{\varkappa\varkappa'}\right)\theta(p_1)\\ &\quad\times\int\cdots\int dp_5\ldots dp_n\, e^{i\sum\limits_{k=5}^{n} p_k x_k} \mathop{\Pi}_{\substack{\lambda=1\ldots 4\\ k=5\ldots n}}\delta\left(p_k p_\lambda + a_{k\lambda}\right) \mathop{\Pi}_{k_1\le k_2=5}^{n}\delta\left(p_{k_1}p_{k_2} + a_{k_1k_2}\right)\\ &= \frac{(-i)^{n-4}}{(2\pi)^{3(n-4)}}(-D)^{\frac{(n-4)^2}{2}} \mathop{\Pi}_{k_1\le k_2=5}^{n}\delta\left(Da_{k_1k_2} - \sum_{\lambda,\lambda'=1}^{4} a_{k_1\lambda}\Delta_{\lambda\lambda'}a_{\lambda' k_2}\right)\Delta_5^{(+)}(y_\varkappa; a_{\varkappa\varkappa'})\mathop{\Pi}_{k=5}^{n}\theta(a_{1k}), \end{aligned}\right\} \tag{15}$$

where

$$y_\varkappa = x_\varkappa + \sum_{\lambda=1}^{4}\sum_{k=5}^{n}\frac{\Delta_{\varkappa\lambda}a_{\lambda k}x_k}{D}. \tag{15a}$$

Note that the indices $\varkappa$ and $\varkappa'$ in $\Delta_5^{(+)}(y_\varkappa; a_{\varkappa\varkappa'})$ only take values from 1 to 4. Eq. (15) shows explicitly how the general function $\Delta_{n+1}^{(+)}(x;a)$ can be expressed as $\Delta_5^{(+)}(y;a)$

multiplied by certain factors. The $\delta$-functions appearing in (15) are an expression of the fact that the vectors $p_5 \ldots p_n$ are linearly dependent on the vectors $p_1 \ldots p_4$.

This result tells us that apart from the two functions exhibited in Eqs. (5) we have to discuss only two more functions, viz. $\Delta_4^{(+)}$ and $\Delta_5^{(+)}$, which somewhat simplifies our task.

## II. Connection Between the Functions $\Delta_{n+1}^{(+)}$ and $\Delta_5^{(+)}$ when $n < 4$

In the preceding section we have demonstrated how one can express every function $\Delta_n^{(+)}$ with $n > 5$ in terms of the function $\Delta_5^{(+)}$. The mathematical reason for this reduction was the fact that space time has four dimensions and therefore more than four vectors must be linearly dependent. We now want to remark that one can also express $\Delta_n^{(+)}$ with $n < 5$ in terms of $\Delta_5^{(+)}$ with one or more of the vectors $x$ put equal to zero. In fact, if we put e. g. $x_4 = 0$, the result of the $p_4$ integration can only depend on the masses $a_{kl}$. This follows from simple considerations of Lorentz invariance. Therefore, the function $\Delta_5^{(+)}$ with $x_4 = 0$ must be equal to the function $\Delta_4^{(+)}$ of $x_1$, $x_2$ and $x_3$ multiplied by a certain function of the masses. To see this in detail we specialize the definition (4a) by putting $x_4 = 0$ in it and obtain

$$\Delta_5^{(+)}\Big|_{x_4=0} = \frac{(-i)^4}{(2\pi)^{12}} \int \ldots \int dp_1 \ldots dp_3 \, e^{i \sum_{k=1}^{3} p_k x_k} \prod_{k \leq l < 4} \delta(p_k p_l + a_{kl}) \, \theta(p_1) \prod_{k=2}^{4} \theta(a_{1k}) \, I_4, \tag{16}$$

where

$$I_4 = \int dp_4 \, \delta(p_4^2 + a_{44}) \prod_{k=1}^{3} \delta(p_4 p_k + a_{4k}) = \frac{1}{\sqrt{-D}} \theta(-D), \tag{16a}$$

with the determinant $D$ defined by Eq. (9a). We assume explicitly that this determinant does not vanish. Otherwise, the function $\Delta_5^{(+)}$ does not exist.

We thus obtain the following connection between the functions $\Delta_4^{(+)}$ and $\Delta_5^{(+)}$:

$$\Delta_5^{(+)}\Big|_{x_4=0} = \frac{-i}{(2\pi)^3} \frac{1}{\sqrt{-D}} \theta(-D) \, \theta(a_{41}) \, \Delta_4^{(+)}(x; a). \tag{17}$$

The function $\Delta_4^{(+)}$ in (17) depends only on the vectors $x_1$, $x_2$ and $x_3$ and on the masses $a_{kl}$ with $k$ and $l$ both $< 4$.

Relation (17) enables us to find the function $\Delta_4^{(+)}$ directly from $\Delta_5^{(+)}$. Therefore, the only function to be discussed before we know $\Delta_{n+1}^{(+)}$ for all values of $n$ is $\Delta_5^{(+)}$.

However, it will be useful to consider also the special case of $\Delta_5^{(+)}$ when two of the vectors $x_1 \ldots x_4$ vanish, e. g. $x_3 = x_4 = 0$. From (4a) we then find

$$\left.\begin{aligned} \varDelta_5^{(+)}\Big|_{x_3=x_4=0} = \frac{(-i)^4}{(2\pi)^{12}} \iint dp_1\, dp_2\, e^{i\sum_{k=1}^{2} p_k x_k} \delta(p_1^2+a_{11})\,\delta(p_2^2+a_{22})\,\delta(p_1 p_2+a_{12}) \\ \times\, \theta(p_1) \prod_{k=2}^{4} \theta(a_{1k})\, I_{34}, \end{aligned}\right\} \quad (18)$$

where

$$\left.\begin{aligned} I_{34} &= \iint dp_3\, dp_4\, \delta(p_3^2+a_{33})\,\delta(p_4^2+a_{44}) \prod_{k=1}^{3} \delta(p_4 p_k + a_{4k}) \prod_{l=1}^{2} \delta(p_3 p_l + a_{3l}) \\ &= \frac{1}{\sqrt{-D}}\,\theta(-D) \int dp_3\, \delta(p_3^2+a_{33})\,\delta(p_1 p_3+a_{13})\,\delta(p_2 p_3+a_{23}) \\ &= \frac{\pi}{\sqrt{a_{12}^2-a_{11}a_{22}}}\,\frac{1}{\sqrt{-D}}\,\theta(-D)\,\theta(D_0), \end{aligned}\right\} \quad (18\,a)$$

with

$$D_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}. \quad (18\,b)$$

The integral $I_{34}$ does not exist if $a_{12}^2 \leq a_{11}\,a_{22}$. When the determinant $D$ does not vanish, it is always possible to relabel the vectors $p_k$ in such a way that the determinant $D_0$ and the expression $a_{12}^2 - a_{11}\,a_{22}$ are different from zero. For simplicity, we assume this relabelling to be made in (18).

From Eqs. (18) we now obtain the following connection between the functions $\varDelta_3^{(+)}$ and $\varDelta_5^{(+)}$:

$$\varDelta_5^{(+)}\Big|_{x_3=x_4=0} = -\frac{\pi}{(2\pi)^6}\cdot\frac{1}{\sqrt{a_{12}^2-a_{11}\,a_{22}}}\cdot\frac{1}{\sqrt{-D}}\,\theta(-D)\,\theta(D_0)\,\theta(a_{14})\,\theta(a_{13})\,\varDelta_3^{(+)}(x_1, x_2; a_{11}, a_{22}, a_{12}). \quad (19)$$

In fact, Eq. (19) will turn out to be a very useful tool in our later discussion of the function $\varDelta_5^{(+)}$.

For completeness, we want to mention that one can also get a relation between the functions $\varDelta_5^{(+)}$ and $\varDelta_2^{(+)}$ similar to Eq. (19). However, this relation will not be used in the following, and we do not want to give it explicitly.

We have now established all connections needed to determine the general function $\varDelta_{n+1}^{(+)}$ for an arbitrary value of $n$ from the function $\varDelta_5^{(+)}$. The following discussion is therefore devoted entirely to this function.

## III. Transformation of the Function $\Delta_5^{(+)}$ to a Standard Form

We now turn to a discussion of the invariance property of the function $\Delta_5^{(+)}$ mentioned at the end of the introduction. For this purpose we remark that we can make a transformation of the variables $p_k$ in the definition (4a) with a non-singular real matrix $\Lambda_{kl}$ in the following way:

$$q_k = \sum_{l=1}^{4} \Lambda_{kl}\, p_l; \quad Det\,|\Lambda_{kl}| \neq 0. \tag{20}$$

Note that this is not a Lorentz transformation where the components of each vector are transformed among themselves, but a transformation among the vectors where all components are transformed in the same way. The scalar products of the vectors $q_k$ are given by

$$q_k\, q_l + a'_{kl} = 0, \tag{21 a}$$

with

$$a'_{kl} = \sum_{k', l'=1}^{4} \Lambda_{kk'}\, a_{k'l'}\, \Lambda_{ll'}. \tag{21 b}$$

If we introduce a matrix notation and write $A$ for the matrix $a_{kl}$, Eq. (21 b) can be written in the condensed form

$$A' = \Lambda\, A\, \Lambda^T. \tag{21 c}$$

By elementary considerations one finds the following two formulae:

$$dp_1 \ldots dp_4 = (Det\,|\Lambda|)^{-4}\, dq_1 \ldots dq_4, \tag{22 a}$$

$$\prod_{k \leq l=1}^{4} \delta(p_k\, p_l + a_{kl}) = (Det\,|\Lambda|)^5 \prod_{k \leq l=1}^{4} \delta(q_k\, q_l + a'_{kl}). \tag{22 b}$$

Therefore, we can write the integral $\Delta_5^{(+)}$ as

$$\Delta_5^{(+)} = \frac{Det\,|\Lambda|}{(2\,\pi)^{12}} \int \ldots \int dq_1 \ldots dq_4\, e^{i\sum_{k=1}^{4} y_k q_k} \prod_{k \leq l=1}^{4} \delta(q_k\, q_l + a'_{kl}) \prod_{k=2}^{4} \theta(a_{1k})\, \theta((\Lambda^{-1} q)_1), \tag{23}$$

with

$$y_k = \sum_{l=1}^{4} x_l\, (\Lambda^{-1})_{lk}. \tag{23 a}$$

Apart from the factor $Det\,|\,\Lambda\,|$ in front and the behaviour of the step functions $\theta(p_k)$, the function $\Delta_5^{(+)}$ is therefore invariant under the combined transformations (20) and (23a)[8].

We now specialize the general matrix $\Lambda_{kl}$ to the following:

$$q_k = p_k - \sum_{l=1}^{k-1} c_{kl}\, p_l\,. \tag{24}$$

Provided the vectors $p_k$ are labelled as indicated in connection with equation (18), we can determine the constants $c_{kl}$ in (24) in such a way that all the non-diagonal products $q_k\, q_l = a'_{kl}$ vanish. A simple consideration shows that the diagonal masses $a'_{kk}$ for that case are given by

$$a'_{11} = a_{11}\,, \tag{25a}$$

$$a'_{22} = \frac{a_{11}\, a_{22} - a_{12}^2}{a_{11}}\,, \tag{25b}$$

$$a'_{33} = \frac{D_0}{a_{11}\, a_{22} - a_{12}^2}\,, \tag{25c}$$

$$a'_{44} = \frac{D}{D_0}\,. \tag{25d}$$

As $Det\,|\,\Lambda\,| = 1$ for the transformation (24), we find from (23)

$$\Delta_5^{(+)} = \frac{1}{(2\,\pi)^{12}} \prod_{k=2}^{4} \theta(a_{1k}) \int \cdots \int dq_1 \ldots dq_4\, e^{i\sum_{k=1}^{4} y_k q_k} \prod_{k=1}^{4} \delta(q_k^2 + a'_{kk}) \prod_{k<l} \delta(q_k\, q_l)\, \theta(q_1)\,. \tag{26}$$

The vector $q_1$ is the same as the vector $p_1$ and therefore timelike. As all the other vectors $q_k$ are orthogonal to the vector $q_1$, it follows that the masses $a'_{kk}$ in (25) for $k \neq 1$ must be negative. We then make a new transformation changing all the vectors $q_k$ by constant positive factors so as to make the absolute value of their squares equal to one. This yields

[8] In a space with a Euclidean metric instead of the Lorentz metric, integrals corresponding to our $\Delta_5^{(+)}$ but defined without the step functions $\theta(p_k)$ are intimately related to the generalized Bessel functions studied by S. Bochner: Med. Lunds Univ. Mat. Sem. Suppl. (1952), p. 12. A symmetry property of these Bessel functions corresponding to the invariance of our functions under the transformation (20) and (23a) has also been mentioned by Bochner in his paper. For the functions $\Delta_{n+1}^{(+)}(x;a)$ studied here, the same result was stated and proved in the thesis (unpublished) of D. Hall (Princeton 1956). This paper also contains the remark that all these functions with $n > 4$ can be reduced to what is here called $\Delta_5^{(+)}(x;a)$. We are indebted to Professor A. Wightman for making this thesis available to us.

$$\left.\begin{aligned} \Delta_5^{(+)} &= \frac{1}{\sqrt{-D}}\,\theta(-D)\,\theta(D_0)\,\theta(a_{12}^2 - a_{11}a_{22})\prod_{k=2}^{4}\theta(a_{1k}) \\ &\times \frac{1}{(2\pi)^{12}}\int\cdots\int dq_1\ldots dq_4\, e^{i\sum_{k=1}^{4} y_k q_k}\,\delta(q_1^2+1)\,\theta(q_1)\prod_{k=2}^{4}\delta(q_k^2-1)\prod_{k<l}\delta(q_k q_l), \end{aligned}\right\} \tag{27}$$

where the vectors $y_k$ are given by

$$y_k = \sqrt{|a'_{kk}|}\sum_{l=1}^{4} x_l\,(\Lambda^{-1})_{lk} \tag{27a}$$

with the matrix $\Lambda$ of Eq. (24).

The transformation (24) is not the only one that will make all the non-diagonal masses equal to zero. In fact, we can make a further transformation among the $q_k$'s in (27) with a new real matrix $\Lambda'$ that leaves the quadratic form $q_1^2 - q_2^2 - q_3^2 - q_4^2$ invariant and with determinant $+1$. With such a transformation the matrix $y_k y_l$ can in many cases be made diagonal. We are then left with the integral

$$\Delta_5^{(+)} = \frac{1}{(2\pi)^{12}}\frac{1}{\sqrt{-D}}\,\theta(-D)\,\theta(D_0)\,\theta(a_{12}^2 - a_{11}a_{22})\prod_{k=2}^{4}\theta(a_{1k})\, I, \tag{28a}$$

$$I = \int\cdots\int dp_1\ldots dp_4\, e^{i\sum_{k=1}^{4} p_k x_k}\,\delta(p_1^2+1)\,\theta(p_1)\prod_{k=2}^{4}\delta(p_k^2-1)\prod_{k<l}\delta(p_k p_l). \tag{28b}$$

In the expression (28b) the vectors $x_k$ are all orthogonal to each other. The squares of these vectors are determined by the eigenvalues $\sigma_1 \ldots \sigma_4$ of the matrix $Y$ defined by

$$\left.\begin{aligned} Y_{kl} &= y_k y_l; \quad k \neq 1, \quad l \neq 1, \\ Y_{1k} &= i y_1 y_k; \quad k \neq 1, \\ Y_{11} &= -y_1^2. \end{aligned}\right\} \tag{29}$$

We have

$$x_1^2 + \sigma_1 = 0; \quad x_k^2 - \sigma_k = 0, \quad k \neq 1. \tag{30}$$

According to (27a), the matrix $Y$ can be written

$$Y = \sqrt{-A'}\,\Lambda^{-1^T} X \Lambda^{-1}\sqrt{-A'}. \tag{31}$$

The elements $X_{kl}$ of the matrix $X$ in (31) are given by the scalar products $x_k x_l$ of the *original* vectors $x_k$ in (4a). $\sqrt{-A'}$ is a diagonal matrix with matrix elements given by $i\sqrt{a_{11}}$ for the first element and by $\sqrt{-a_{kk}}$ for $k \neq 1$. The eigenvalues of $Y$ in (31) are the same as the eigenvalues of the matrix

$$-\Lambda^{-1^T} X \Lambda^{-1} A' = -\Lambda^{-1^T} X \Lambda^{-1} \Lambda A \Lambda^T = -\Lambda^{-1^T} X A \Lambda^T, \tag{32}$$

where $A$ is the matrix of the original masses $a_{kl}$ according to Eq. (21c). From (32) it follows immediately that the quantities $\sigma_k$ in (30) are the eigenvalues of the matrix product $-XA$. This result allows us to compute the $\sigma$'s directly from the quantities appearing in Eq. (4a) without going through all the transformations in detail.

The condition that the matrix $\Lambda'$ transforming (27) into (28) is a real matrix is the condition that all the eigenvalues $\sigma_k$ are real. For given matrices $X$ and $A$ this may or may not be the case. Instead of trying to discuss explicitly the case where $\Lambda'$ is not real, we use the result of ref. 2 that the function $\Delta_5^{(+)}$ is the boundary value of an analytic function of the scalar products $x_k x_l$. For a given matrix $A$, we choose such matrices $X$ that the quantities $\sigma_k$ are all real and compute the function $\Delta_5^{(+)}$ for that case. The value of $\Delta_5^{(+)}$ for some other matrix $X$ can then be obtained from our result by analytic continuation.

## IV. Calculation of the Integral (28b)

The discussion of the previous sections has reduced the computation of all the functions $\Delta_{n+1}^{(+)}$ to the evaluation of the integral (28b) with vectors $x_1 \ldots x_4$ that are orthogonal to each other. Further, we know from the result of ref. 2 that this integral is the boundary value of an analytic function of the squares $x_k^2$. Therefore, it is sufficient to compute it for, say, $x_1$ timelike with positive time component and hence the other $x_k$ spacelike. To make the integral convergent we further assume that $x_{10}$ has a negative imaginary part, which is not necessarily infinitesimal. It is then a question of labelling to suppose $|x_2| \geq |x_3| \geq |x_4| \geq 0$ and to choose the coordinate axes in such a way that $x_2$ lies along the positive $x$-direction, $x_3$ along the positive $y$-direction, and $x_4$ along the positive $z$-direction. In this way we get

$$\left.\begin{aligned} I = \int \cdots \int dp_1 \ldots dp_4 \, e^{-ip_{10}x_{10} + ip_{2x}x_{2x} + ip_{3y}x_{3y} + ip_{4z}x_{4z}} \delta(p_1^2 + 1)\, \theta(p_1) \\ \times \prod_{k=2}^{4} \delta(p_k^2 - 1) \prod_{k<l=2}^{4} \delta(p_k p_l). \end{aligned}\right\} \tag{33}$$

The actual order in which the following integrations are performed is a question of convenience. We have found it convenient to start with the $p_3$ and $p_4$ integrations and the integrations over those components over $p_1$ and $p_2$ that are orthogonal to the plane spanned by $x_1$ and $x_2$. This can also be so formulated that we first compute the integral

$$I_0 = \iint_{(y,z)} dp_1\, dp_2 \iint_{(t,x)} dp_3\, dp_4\, \delta(p_1^2 + 1) \prod_{k=2}^{4} \delta(p_k^2 - 1) \prod_{k<l=2}^{4} \delta(p_k p_l). \tag{34}$$

Note that the integral $I_0$ does not contain any exponential functions and, therefore, can be computed by elementary means. As is shown in Appendix I, the result of this computation is

$$\left.\begin{aligned} I_0 = \frac{\lambda^{\frac{9}{2}}}{A^{\frac{1}{2}}}\,\delta\left((q_1 q_2)^2 + (q_3 q_4)^2 - q_1^2 q_2^2 - q_3^2 q_4^2\right)\delta\left(q_3^2 + q_4^2 - q_2^2 + q_1^2\right) \\ \times\,\theta(A)\,\theta\left(q_1^2(q_2^2 + q_1^2) - 2(q_1 q_2)^2\right), \end{aligned}\right\} \tag{35}$$

$$\lambda = (q_1 q_2)^2 - q_1^2 q_2^2 > 0\,, \tag{35 a}$$

$$A = (q_2^2 - 1)(q_1^2 + 1) - (q_1 q_2)^2. \tag{35 b}$$

In Eq. (35), the vectors $q_k$ are two-dimensional vectors built up of those components of the vectors $p_k$ that have not been integrated over in (34). To be more precise, $q_1$ and $q_2$ have components in the $x$ and $t$ directions, while $q_3$ and $q_4$ have components along the $y$ and $z$ directions.

We next perform the integrations over $q_3$ and $q_4$ and get, according to Appendix II,

$$\left.\begin{aligned} I_1 &= \iint\limits_{(y,\,z)} dq_3\, dq_4\, e^{ip_{3y}x_{3y} + ip_{4z}x_{4z}} \cdot I_0 \\ &= \frac{\pi^2}{2}\frac{\lambda^4}{A^{\frac{1}{2}}}\left\{J_0\left(\frac{x_{3y} + x_{4z}}{2}\sqrt{q_2^2 - q_1^2 + 2\sqrt{\lambda}}\right) J_0\left(\frac{x_{3y} - x_{4z}}{2}\sqrt{q_2^2 - q_1^2 - 2\sqrt{\lambda}}\right)\right. \\ &\qquad \left. + J_0\left(\frac{x_{3y} + x_{4z}}{2}\sqrt{q_2^2 - q_1^2 - 2\sqrt{\lambda}}\right) J_0\left(\frac{x_{3y} - x_{4z}}{2}\sqrt{q_2^2 - q_1^2 + 2\sqrt{\lambda}}\right)\right\} \\ &\quad \times\,\theta(A)\,\theta\left(q_1^2(q_2^2 + q_1^2) - 2(q_1 q_2)^2\right)\theta\left(q_2^2 - q_1^2 - 2\sqrt{\lambda}\right)\theta\left(q_2^2 - q_1^2 + 2\sqrt{\lambda}\right). \end{aligned}\right\} \tag{36}$$

According to Appendix II, the integrations over $q_1$ and $q_2$ can be arranged in such a way that the integral $I$ in (33) reads

$$\left.\begin{aligned} I = \frac{1}{2}\iint\limits_0^\infty dr_1\, dr_2\, F(r_1, r_2)\, H_0^{(1)}\left(-\frac{x_{10} + x_{2x}}{2} r_1\right) H_0^{(1)}\left(-\frac{x_{10} - x_{2x}}{2} r_2\right) \\ \times\left\{J_0\left(\frac{x_{3y} + x_{4z}}{2} r_1\right) J_0\left(\frac{x_{3y} - x_{4z}}{2} r_2\right) + J_0\left(\frac{x_{3y} - x_{4z}}{2} r_1\right) J_0\left(\frac{x_{3y} + x_{4z}}{2} r_2\right)\right\}. \end{aligned}\right\} \tag{36 a}$$

The function $F(r_1, r_2)$ in (36 a) is a certain function of $r_1$ and $r_2$, but independent of the $x_k$'s. This function is given explicitly in Appendix II, but the details of it are not important for the following discussion.

We now make use of the formula[9]

$$H_0^{(1)}(z)\,J_0(Z)=\frac{1}{\pi}\int_0^{\pi}d\psi\,H_0^{(1)}(\mathfrak{z}),\tag{37}$$

$$\mathfrak{z}=\sqrt{z^2+Z^2+2\,z\,Z\cos\psi},\tag{37 a}$$

where the sign of the root in (37 a) is defined so as to make $\mathfrak{z}\to z$ for $|z|\to\infty$. Eq. (37) is valid as long as $|z|>|Z|$, in which case $\mathfrak{z}$ never passes through the origin when $\psi$ goes from 0 to $\pi$. Therefore, it is important in our application to have the absolute value of the argument of the Hankel function larger than the absolute value of the argument of the Bessel function. One way to achieve this is to make the imaginary part of $x_{10}$ sufficiently large and to compute the integral for that case. We can then again use the analyticity properties established in ref. 2 to continue our result to arbitrary values of the vectors involved.

With the aid of (37) we can write (36 a) as

$$I=\frac{1}{2}(I^{(1)}+I^{(2)}),\tag{38 a}$$

$$I^{(1)}=\frac{1}{\pi^2}\iint_0^{\pi}d\psi_1\,d\psi_2\iint_0^{\infty}dr_1\,dr_2\,F(r_1,r_2)\,H_0^{(1)}\left(\frac{1}{2}r_1\mathfrak{z}_1\right)H_0^{(1)}\left(\frac{1}{2}r_2\mathfrak{z}_2\right),\tag{38 b}$$

$$\mathfrak{z}_{1,2}^2=(x_{10}\pm x_{2x})^2+(x_{3y}\pm x_{4z})^2+2\,(x_{10}\pm x_{2x})(x_{3y}\pm x_{4z})\cos\psi_{1,2},\ Im\,\mathfrak{z}_{1,2}>0.\tag{38 c}$$

The term $I^{(2)}$ in (38 a) is obtained from $I^{(1)}$ if we replace e. g. $x_{4z}$ by $-x_{4z}$.

By putting $x_{3y}=x_{4z}=0$ in (36 a), and using the result of Eq. (19) together with Eqs. (5) and (28), we get the formula

$$\left.\begin{aligned}&\iint_0^{\infty}dr_1\,dr_2\,F(r_1,r_2)\,H_0^{(1)}\left(-\frac{x_{10}+x_{2x}}{2}r_1\right)H_0^{(1)}\left(-\frac{x_{10}-x_{2x}}{2}r_2\right)\\&\quad=\frac{i\,\pi^4}{x_{10}\,x_{2x}}\left(H_0^{(1)}(-x_{10}-x_{2x})-H_0^{(1)}(-x_{10}+x_{2x})\right).\end{aligned}\right\}\tag{39}$$

Eq. (39) is correct when the imaginary parts of the arguments of the Hankel functions are positive. Otherwise, the integral on the left-hand side is not convergent. Eq. (39) allows us to perform the two $r$ integrations in (38 b) explicitly, yielding the result

$$I^{(1)}=4\,i\,\pi^2\iint_0^{\pi}d\psi_1\,d\psi_2\frac{H_0^{(1)}(\mathfrak{z}_1)-H_0^{(1)}(\mathfrak{z}_2)}{\mathfrak{z}_1^2-\mathfrak{z}_2^2}.\tag{40}$$

[9] Cf. e. g. G. N. Watson: Theory of Bessel Functions, second edition, (Cambridge 1944). Eq. (37) is a special case of formula (16) on page 367 of this reference.

A simple algebraic calculation shows that, if we introduce the argument of each Hankel function as variable of integration instead of one of the angles $\psi$, the integration over the other angle $\psi$ can be made by elementary means and yields

$$I^{(1)} = -8\,i\,\pi^3 \left[ \int_{t_1}^{t_2} \frac{t\,dt\,H_0^{(1)}(t)}{\sqrt{-P_1(t)}\cdot\sqrt{P_2(t)}} - \int_{t_3}^{t_4} \frac{t\,dt\,H_0^{(1)}(t)}{\sqrt{P_1(t)}\cdot\sqrt{-P_2(t)}} \right], \tag{41}$$

$$P_1(t) = (t^2 - t_1^2)(t^2 - t_2^2); \qquad P_2(t) = (t^2 - t_3^2)(t^2 - t_4^2), \tag{41 a}$$

$$\left.\begin{aligned} t_1 &= -x_{10} - x_{2x} - x_{3y} - x_{4z}; \quad & t_2 &= -x_{10} - x_{2x} + x_{3y} + x_{4z}; \\ t_3 &= -x_{10} + x_{2x} - x_{3y} + x_{4z}; \quad & t_4 &= -x_{10} + x_{2x} + x_{3y} - x_{4z}. \end{aligned}\right\} \tag{41 b}$$

For the particular case of the vectors $x_k$, which we have investigated, all the numbers $t_k$ have the same positive imaginary part, while the real parts fulfil the inequalities

$$Re\,t_4 \geq Re\,t_3 \geq Re\,t_2 \geq Re\,t_1. \tag{41 c}$$

Finally, all the square roots appearing in (41) are defined to have positive imaginary parts along the path of integration. When $\psi$ goes through real values, the two paths of integration in (41) are two arcs of hyperbolas, as shown in fig. 1.

We can now redefine the square roots by introducing cuts in the complex $t$-plane between the pairs of points $(t_1, t_2)$; $(-t_1, -t_2)$; $(t_3, t_4)$, and $(-t_3, -t_4)$, and defining $\sqrt{P_k(t)}$ to approach $t^2$ for large $|t|$.

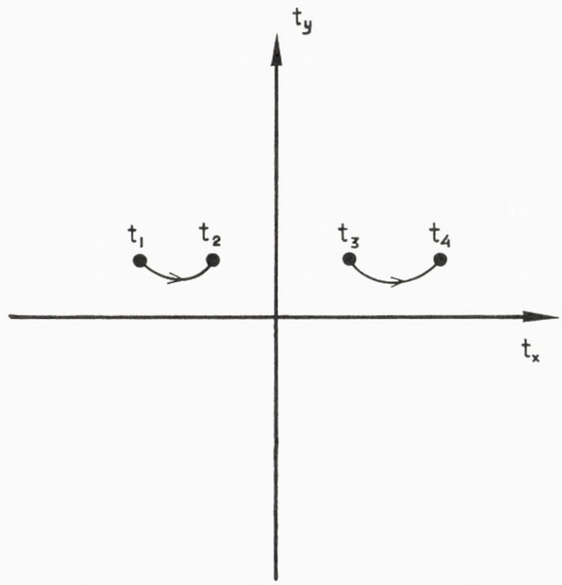


Fig. 1. Paths of integration in Eq. (41).

As shown in Appendix III, this new definition of the square roots implies that $\sqrt{P_1(t)}$ is the same as in Eq. (41), while the sign in front of $\sqrt{P_2(t)}$ has to be changed. Further, one has to replace $\sqrt{-P_k(t)}$ by $i\sqrt{P_k(t)}$ for $k=1$ as well as for $k=2$. This gives

$$\left.\begin{aligned} I^{(1)} &= (2\pi)^3 \left[\int_{t_1}^{t_2} \frac{t\,dt\,H_0^{(1)}(t)}{\sqrt{P_1(t)}\sqrt{P_2(t)}} + \int_{t_3}^{t_4} \frac{t\,dt\,H_0^{(1)}(t)}{\sqrt{P_1(t)}\sqrt{P_2(t)}}\right] \\ &= \frac{(2\pi)^3}{2}\int_{-\infty}^{\infty} \frac{t\,dt\,H_0^{(1)}(t)}{\sqrt{\Xi(t)}}, \end{aligned}\right\} \quad (42)$$

$$\Xi(t) = P_1(t)\,P_2(t). \quad (42\,a)$$

$\sqrt{\Xi(t)}$ in Eq. (42) is defined with the aid of the cuts mentioned earlier and the condition that it approaches $t^4$ for large values of $|t|$. A straightforward algebraic calculation allows us to express the polynomial $\Xi(t)$ in terms of the squares of the vectors $x_k$, i. e., in terms of the eigenvalues $\sigma_k$ mentioned in Eq. (30). The result of this calculation is

$$\Xi(t) = \left[(t^2-Q)^2 - R + \sqrt{T}\right]^2 - t^2 S - 8\sqrt{T}\,t^2(t^2-Q), \quad (43\,a)$$

$$Q = \sigma_1+\sigma_2+\sigma_3+\sigma_4 = I_1, \quad (43\,b)$$

$$R = 4(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4) = 2(I_1^2 - I_2), \quad (43\,c)$$

$$S = 64(\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4) = \frac{32}{3}(I_1^3 - 3I_1I_2 + 2I_3), \quad (43\,d)$$

$$T = 64\,\sigma_1\sigma_2\sigma_3\sigma_4 = \frac{8}{3}(I_1^4 - 6I_1^2I_2 + 3I_2^2 + 8I_3I_1 - 6I_4), \quad (43\,e)$$

$$I_k = Sp\left[(AX)^k\right]; \quad k = 1\ldots 4, \quad (43\,f)$$

where the matrices $A$ and $X$ are defined in connection with Eq. (32). Note that $T$ is the product of the two determinants of the matrices $X$ and $A$ (apart from a numerical factor), $S$ is a sum of products of $3\times 3$ subdeterminants of $X$ and $A$, while $R$ is a sum of products of $2\times 2$ subdeterminants from the same matrices. This means that $T=0$ if the vectors $x_k$ lie in a three-plane, $S=T=0$ if they lie in a two-plane and $S=T=R=0$ if they are all collinear.

We now finish with the remark that $I^{(2)}$ in (38a) is obtained from $I^{(1)}$ in (42) if we replace $\sqrt{T}$ in (43a) by $-\sqrt{T}$. It follows that

$$\left.\begin{aligned}\Delta_5^{(+)} = \frac{1}{(2\pi)^9}\frac{1}{4\sqrt{-D}}\int_{-\infty}^{\infty} t\,dt\,H_0^{(1)}(t)\left\{\frac{1}{\sqrt{\Xi_1}}+\frac{1}{\sqrt{\Xi_2}}\right\}\theta(-D)\,\theta(D_0)\\ \times\theta(a_{12}^2-a_{11}a_{22})\prod_{k=2}^{4}\theta(a_{1k}),\end{aligned}\right\} \tag{44}$$

$$\Xi_{1,2} = \left[(t^2-Q)^2-R\pm\sqrt{T}\right]^2-t^2 S\mp 8\sqrt{T}\,t^2(t^2-Q), \tag{44a}$$

where the determinants $D$ and $D_0$ are defined in Eqs. (9a) and (18b).

The result (44) has been derived under somewhat special assumptions about the matrix $X$, but it follows from the analyticity properties mentioned several times earlier that it is valid for all $X$. Therefore, it yields the desired result. In particular, the two equations

$$\Xi_{1,2} = 0 \qquad \text{for} \qquad 0 \leq t^2 < \infty \tag{45}$$

are the desired generalization of Eq. (6) for the function $\Delta_{n+1}^{(+)}$ with $4 \geq n > 2$. Note that (45) is reduced to (6) when $S = T = 0$.

## V. A Simplified Version of the Result of Section I

The result of Section I allows us to express all functions $\Delta_{n+1}^{(+)}$ with $n > 4$ in terms of $\Delta_5^{(+)}$ with the aid of the formula (15). An explicit application of this formula is rather involved as one has to compute the vectors $y_\varkappa$ with the aid of (15a). In this section, we want to give an explicit expression for the quantities $Q$, $R$, $S$, and $T$ in terms of the original vectors $x_k$ when $n > 4$. To this purpose we write Eq. (15a) as

$$y = x + M\,\xi, \tag{46}$$

where $y$ and $x$ are $1\times 4$ matrices, $\xi$ is the $(n-4)\times 1$ matrix $(x_5; x_6; \ldots x_n)$, and $M$ is a $4\times(n-4)$ matrix defined by

$$M_{\varkappa k} = \frac{1}{D}\sum_{\lambda=1}^{4}\Delta_{\varkappa\lambda}\,a_{\lambda k}; \quad \varkappa = 1\ldots 4; \quad k = 5\ldots n. \tag{47}$$

For convenience, we split the matrix $A$ in the following way:

$$A = \left(\begin{array}{c|c} A_0 & A_1 \\ \hline A_1^T & A_2 \end{array}\right), \tag{48}$$

where $A_0$ is a $4 \times 4$ matrix with matrix element $a_{\varkappa\lambda}$ ($\varkappa, \lambda = 1 \ldots 4$), $A_1$ is a $4 \times (n-4)$ matrix with matrix elements $a_{\varkappa k}$ ($\varkappa = 1 \ldots 4, k = 5 \ldots n$), etc. In this matrix notation we can write Eq. (13) as

$$A_0 M = A_1 . \tag{49}$$

From the $\delta$-functions in (15) it also follows that

$$A_1^T M = A_2 . \tag{50}$$

Eqs. (49) and (50) can be combined to yield

$$M^T A_0 M = A_2 . \tag{51}$$

We now consider the tensor $F_{\mu\nu}$ with two vector indices $\mu$ and $\nu$ in ordinary space time defined by

$$F_{\mu\nu} = \sum_{\varkappa,\lambda=1}^{4} (y_\varkappa)_\mu a_{\varkappa\lambda} (y_\lambda)_\nu \equiv y_\mu^T A_0 y_\nu . \tag{52}$$

With the aid of (46), (49), and (51) we get

$$\left.\begin{aligned} F_{\mu\nu} &= (\xi_\mu^T M^T + x_\mu^T) A_0 (x_\nu + M \xi_\nu) \\ &= x_\mu^T A_0 x_\nu + x_\mu^T A_1 \xi_\nu + \xi_\mu^T A_1^T x_\nu + \xi_\mu^T A_2 \xi_\nu \\ &= \sum_{k,l=1}^{n} (x_k)_\nu a_{kl} (x_l)_\mu . \end{aligned}\right\} \tag{53}$$

The important quantity in the formulae (43) is the matrix $G_{\varkappa\lambda}^{(y)}$ defined by

$$G_{\varkappa\lambda}^{(y)} = \sum_{\mu=1}^{4} \sum_{\varkappa'=1}^{4} (y_\varkappa)_\mu (y_{\varkappa'})_\mu a_{\varkappa'\lambda} \equiv (Y A_0)_{\varkappa\lambda}; \quad \varkappa, \lambda = 1 \ldots 4, \tag{54}$$

and the quantities $Q \ldots T$ are obtained from traces of various powers of $G^{(y)}$. However, from (52) and (53) follows, e. g.,

$$Sp\,[G^{(y)}] = \sum_{\mu=1}^{4} F_{\mu\mu} = Sp\,[G^{(x)}], \tag{55}$$

with

$$G_{kl}^{(x)} = \sum_{\mu=1}^{4} \sum_{k'=1}^{n} (x_k)_\mu (x_{k'})_\mu a_{k'l} = (XA)_{kl}; \quad k, l = 1 \ldots n. \tag{55a}$$

In a similar way, one finds

$$I_k = Sp\,[(G^{(y)})^k] = Sp\,[(G^{(x)})^k]; \quad k = 1 \ldots 4. \tag{56}$$

Eq. (56) allows us to compute the quantities $I_k$ and, hence, $Q\dots T$ directly from the given matrices $X$ and $A$ without going through the intermediate steps of computing the $Y$. This is the desired result. Furthermore, we remark that Eq. (56) is formally identical with (43f). Therefore we can consider Eqs. (43) as the general definition of $Q\dots T$ for all values of $n$.

## Appendix I

Consider the integral $I_0$ defined in Eq. (34)

$$\left.\begin{aligned} I_0 = \iint\limits_{(y,z)} d^2 u_1\, d^2 u_2 \iint\limits_{(t,x)} d^2 u_3\, d^2 u_4\, \delta(u_1^2 + 1 + q_1^2) \prod_{k=2}^{4} \delta(u_k^2 - 1 + q_k^2)\, \delta(u_1 u_2 + q_1 q_2) \\ \times \delta(u_3 u_4 + q_3 q_4)\, \delta(u_1 q_3 + u_3 q_1)\, \delta(u_1 q_4 + u_4 q_1)\, \delta(u_2 q_3 + u_3 q_2)\, \delta(u_2 q_4 + u_4 q_2)\,. \end{aligned}\right\} \quad (A.1)$$

In Eq. (A. 1), we have written $p_k = u_k + q_k$, where $q_k$ are those components of the vectors $p_k$ over which we do not integrate (cf. the remark after Eq. (35)).

With the aid of a transformation similar to that in (24) we first introduce unit vectors $e_k$ orthogonal to each other in the following way:

$$\left.\begin{aligned} e_1 &= \frac{q_1}{\sqrt{-q_1^2}}; & e_3 &= \frac{q_3}{\sqrt{q_3^2}}; \\ e_2 &= \frac{q_2\sqrt{-q_1^2} + \dfrac{q_1 q_2}{\sqrt{-q_1^2}}\, q_1}{\sqrt{\lambda_{12}}}; & e_4 &= \frac{q_4\sqrt{q_3^2} - \dfrac{q_3 q_4}{\sqrt{q_3^2}}\, q_3}{\sqrt{\lambda_{34}}}, \end{aligned}\right\} \quad (A.2)$$

where $\lambda_{12} = (q_1 q_2)^2 - q_1^2 q_2^2$ and $\lambda_{34} = q_3^2 q_4^2 - (q_3 q_4)^2$. If we simultaneously introduce new variables of integration according to

$$\left.\begin{aligned} v_1 &= \frac{u_1}{\sqrt{-q_1^2}}; & v_3 &= \frac{u_3}{\sqrt{q_3^2}}; \\ v_2 &= \frac{u_2\sqrt{-q_1^2} + \dfrac{q_1 q_2}{\sqrt{-q_1^2}}\, u_1}{\sqrt{\lambda_{12}}}; & v_4 &= \frac{u_4\sqrt{q_3^2} - \dfrac{q_3 q_4}{\sqrt{q_3^2}}\, u_3}{\sqrt{\lambda_{34}}}, \end{aligned}\right\} \quad (A.3)$$

the integral $I_0$ can be written as

$$\left.\begin{aligned} I_0 = \sqrt{\lambda_{12}}\sqrt{\lambda_{34}}\iint\limits_{(y,z)} dv_1\, dv_2 \iint\limits_{(x,t)} dv_3\, dv_4 \prod_{k=1}^{4} \delta(v_k^2 + \varrho_k)\, \delta(v_1 v_2 + \eta_{12})\, \delta(v_3 v_4 + \eta_{34}) \\ \times \delta(v_1 e_3 + v_3 e_1)\, \delta(v_2 e_3 + v_3 e_2)\, \delta(v_1 e_4 + v_4 e_1)\, \delta(v_2 e_4 + v_4 e_1), \end{aligned}\right\} \quad (A.4)$$

with

$$\varrho_1 = \frac{1+q_1^2}{-q_1^2}; \quad \varrho_2 = 1 + \frac{(q_1 q_2)^2 - (q_1^2)^2}{-q_1^2 \lambda_{12}}; \quad \varrho_3 = \frac{q_3^2 - 1}{q_3^2}; \quad \varrho_4 = 1 - \frac{(q_3 q_4)^2 + (q_3^2)^2}{q_3^2 \lambda_{34}} \quad (A.4a)$$

and

$$\eta_{12} = \frac{-1}{\sqrt{\lambda_{12}}} \frac{q_1 q_2}{q_1^2}; \quad \eta_{34} = \frac{1}{\sqrt{\lambda_{34}}} \frac{q_3 q_4}{q_3^2}. \quad (A.4b)$$

As the space of the vectors $e_3$ and $e_4$ has a positive definite metric, we can perform an ordinary rotation in this space and make the vectors $v_3$ and $v_4$ orthogonal to each other. (A corresponding real rotation in the space of $e_1$ and $e_2$ is possible sometimes, but not always.) In this way, we get a formula similar to Eq. (A. 4), except that $\eta_{34} = 0$ and $\varrho_3$ and $\varrho_4$ are replaced by two other quantities $\varrho_3'$ and $\varrho_4'$ determined from

$$\varrho_3' + \varrho_4' = \varrho_3 + \varrho_4 = 2 - \frac{q_3^2 + q_4^2}{\lambda_{34}}, \quad (A.4c)$$

$$\varrho_3' \varrho_4' = \varrho_3 \varrho_4 - \eta_{34}^2 = \frac{(1-q_3^2)(1-q_4^2) - (q_3 q_4)^2}{\lambda_{34}}. \quad (A.4d)$$

If we now introduce a coordinate system with the vector $e_1$ along the time axis, the vector $e_2$ along the $x$-axis, etc., the last four delta functions in Eq. (A. 4) read

$$\delta(v_{1y} - v_{30})\, \delta(v_{2y} + v_{3x})\, \delta(v_{1z} - v_{40})\, \delta(v_{2z} + v_{4x}). \quad (A.5)$$

These four delta functions permit us to make, e. g., the $v_1$ and $v_2$ integrations trivially, yielding

$$\left.\begin{aligned} I_0 = \sqrt{\lambda_{12}}\sqrt{\lambda_{34}} \iint\limits_{(x,t)} dv_3\, dv_4\, \delta(v_3^2 + \varrho_3')\, \delta(v_4^2 + \varrho_4')\, \delta(v_3 v_4) \\ \times \delta(v_{30}^2 + v_{40}^2 + \varrho_1)\, \delta(v_{3x}^2 + v_{4x}^2 + \varrho_2)\, \delta(-v_{30} v_{3x} - v_{40} v_{4x} + \eta_{12}). \end{aligned}\right\} \quad (A.6)$$

The last delta function in (A. 6) can now be written as

$$\left.\begin{aligned} \delta(\eta_{12} - v_{30} v_{3x} - v_{40} v_{4x}) = 2\,|\,\eta_{12}\,|\,\delta(\eta_{12}^2 - (v_{30} v_{3x} + v_{40} v_{4x})^2)\, \theta(\eta_{12}(v_{30} v_{3x} + v_{40} v_{4x})) \\ = 2\,|\,\eta_{12}\,|\,\delta(\eta_{12}^2 - (\varrho_1 + \varrho_3')(\varrho_1 + \varrho_4'))\, \theta(\eta_{12}(v_{30} v_{3x} + v_{40} v_{4x})). \end{aligned}\right\} \quad (A.7)$$

In a similar way, we write

$$\delta(v_{3x}^2 + v_{4x}^2 + \varrho_2) = \delta(\varrho_2 - \varrho_1 - \varrho_3' - \varrho_4'). \tag{A. 8}$$

This gives

$$\left.\begin{aligned} I_0 &= 2\,|\,\eta_{12}\,|\sqrt{\lambda_{12}}\sqrt{\lambda_{34}}\,\delta(\eta_{12}^2 - (\varrho_1 + \varrho_3')(\varrho_1 + \varrho_4'))\,\delta(\varrho_2 - \varrho_1 - \varrho_3' - \varrho_4') \\ &\times \iint dv_3\,dv_4\,\delta(v_3^2 + \varrho_3')\,\delta(v_4^2 + \varrho_4')\,\delta(v_3 v_4)\,\delta(v_{30}^2 + v_{40}^2 + \varrho_1)\,\theta(\eta_{12}(v_{30}v_{3x} + v_{40}v_{4x})). \end{aligned}\right\} \tag{A. 9}$$

The remaining integrations in (A. 9) can now be made by straightforward methods, yielding

$$I_0 = \frac{\sqrt{\lambda_{12}}\sqrt{\lambda_{34}}}{\sqrt{-\varrho_3'\varrho_4'}}\,\delta(\varrho_2 - \varrho_1 - \varrho_3' - \varrho_4')\,\delta(\eta_{12}^2 - (\varrho_1 + \varrho_3')(\varrho_1 + \varrho_4'))\,\theta(-\varrho_3'\varrho_4')\,\theta(\varrho_1 + \varrho_2). \tag{A. 10}$$

If we introduce the expressions (A. 4) for $\varrho_1, \ldots \varrho_4'$ and $\eta_{12}$, we get after some simple manipulations

$$I_0 = \frac{(\lambda_{12})^{\frac{3}{2}}}{\sqrt{A}}\,\delta(\lambda_{12} - \lambda_{34})\,\delta(q_3^2 + q_4^2 - q_2^2 + q_1^2)\,\theta(A)\,\theta(q_1^2(q_1^2 + q_2^2) - 2(q_1 q_2)^2), \tag{A. 11}$$

$$A = (q_2^2 - 1)(q_1^2 + 1) - (q_1 q_2)^2. \tag{A. 11 a}$$

Eq. (A. 11) is identical with Eq. (35) in the main text.

## Appendix II

With the aid of the result of Appendix I, the integral $I_1$ in Eq. (36) can be evaluated by straightforward techniques. We write the definition of this integral in the following way:

$$I_1 = \frac{\lambda^{\frac{3}{2}}}{\sqrt{A}}\,\theta(A)\,\theta(q_1^2(q_2^2 + q_1^2) - 2(q_1 q_2)^2)\,\boldsymbol{I}, \tag{A. 12}$$

$$\boldsymbol{I} = \iint_{(y,z)} dq_3\,dq_4\,e^{iq_{3y}x_{3y} + iq_{4z}x_{4z}}\,\delta(q_3^2 + q_4^2 - \varrho^2)\,\delta(q_3^2 q_4^2 - (q_3 q_4)^2 - \lambda), \tag{A. 12 a}$$

$$\varrho^2 = q_2^2 - q_1^2; \qquad \lambda = (q_1 q_2)^2 - q_1^2 q_2^2. \tag{A. 12 b}$$

The argument of the last delta function is simplified if we introduce a new vector $q_4'$ with the same length as $q_4$, but orthogonal to it, i. e.

$$\left.\begin{aligned} q'_{4y} &= q_{4z}, \\ q'_{4z} &= -q_{4y}. \end{aligned}\right\} \quad \text{(A. 13)}$$

With this definition we get

$$q_4'^2 = q_4^2, \quad \text{(A. 13a)}$$

$$(q_4' q_3)^2 = q_3^2 q_4^2 - (q_3 q_4)^2 = \lambda, \quad \text{(A. 13b)}$$

and write the integral $\boldsymbol{I}$ as

$$\boldsymbol{I} = \frac{1}{2\sqrt{\lambda}} \iint dq_3\, dq_4'\, e^{iq_{3y}x_{3y} + iq'_{4y}x_{4z}}\, \delta(q_3^2 + q_4'^2 - \varrho^2)\left[\delta(q_3 q_4' + \sqrt{\lambda}) + \delta(q_3 q_4' - \sqrt{\lambda})\right]. \quad \text{(A. 14)}$$

If we now introduce the two vectors $p_3 = q_3 + q_4'$ and $p_4 = q_3 - q_4'$ as variables of integration, we get

$$\left.\begin{aligned} \boldsymbol{I} &= \frac{1}{2\sqrt{\lambda}} \iint dp_3\, dp_4\, e^{ip_{3y}\frac{x_{3y}+x_{4z}}{2} + ip_{4y}\frac{x_{3y}-x_{4z}}{2}} \Big\{\delta(p_3^2 - \varrho^2 + 2\sqrt{\lambda})\,\delta(p_4^2 - \varrho^2 - 2\sqrt{\lambda}) \\ &\quad + \delta(p_3^2 - \varrho^2 - 2\sqrt{\lambda})\,\delta(p_4^2 - \varrho^2 + 2\sqrt{\lambda})\Big\} = \frac{\pi^2}{2\sqrt{\lambda}} \Big\{ J_0\left(\frac{x_{3y}+x_{4z}}{2}\sqrt{\varrho^2 - 2\sqrt{\lambda}}\right) \\ &\quad \times J_0\left(\frac{x_{3y}-x_{4z}}{2}\sqrt{\varrho^2 + 2\sqrt{\lambda}}\right) + J_0\left(\frac{x_{3y}+x_{4z}}{2}\sqrt{\varrho^2 + 2\sqrt{\lambda}}\right) J_0\left(\frac{x_{3y}-x_{4z}}{2}\sqrt{\varrho^2 - 2\sqrt{\lambda}}\right)\Big\} \\ &\quad \times \theta(\varrho^2 - 2\sqrt{\lambda})\,\theta(\varrho^2 + 2\sqrt{\lambda}), \end{aligned}\right\} \quad \text{(A. 15)}$$

where $J_0(x)$ is the ordinary Bessel function of order zero. Collecting Eqs. (A. 12) and (A. 15), we get Eq. (36) in the main text.

Essentially the same technique can be used for the integrations over $q_1$ and $q_2$ in (36a). A typical term to be computed is

$$\left.\begin{aligned} \boldsymbol{I}_1 &= \iint_{(x,t)} dq_1\, dq_2\, e^{-iq_{10}x_{10} + iq_{2x}x_{2x}}\, \lambda^4 \frac{\theta(A)}{\sqrt{A}}\, \theta(q_1)\,\theta(q_2^2 - q_1^2 + 2\sqrt{\lambda})\,\theta(q_2^2 - q_1^2 - 2\sqrt{\lambda}) \\ &\quad \times J_0\left(\frac{x_{3y}+x_{4z}}{2}\sqrt{q_2^2 - q_1^2 + 2\sqrt{\lambda}}\right) J_0\left(\frac{x_{3y}-x_{4z}}{2}\sqrt{q_2^2 - q_1^2 - 2\sqrt{\lambda}}\right) \theta(q_1^2(q_1^2 + q_2^2) - 2(q_1 q_2)^2). \end{aligned}\right\} \quad \text{(A. 16)}$$

In this case, we choose $p_1 = q_1 + q_2'$ and $p_2 = q_1 - q_2'$ as variables of integration, where $q_2'$ is defined by

$$\left.\begin{aligned} q'_{2x} &= q_{20}, \\ q'_{20} &= +q_{2x}. \end{aligned}\right\} \quad \text{(A. 17)}$$

This yields

$$q_1'^2 = -q_2^2, \tag{A. 18a}$$

$$(q_2' q_1)^2 = (q_1 q_2)^2 - q_1^2 q_2^2 = \lambda, \tag{A. 18b}$$

and

$$\left.\begin{aligned} \boldsymbol{I}_1 = \iint\limits_{(x,t)} dp_1\, dp_2\, e^{-ip_{10}\frac{x_{10}+x_{2r}}{2} - ip_{20}\frac{x_{10}-x_{2r}}{2}} \left(\frac{p_1^2 - p_2^2}{4}\right)^8 \frac{\theta(A)}{\sqrt{A}}\, \theta(-p_1^2)\, \theta(-p_2^2)\, \theta(p_1 + p_2) \\ \times J_0\left(\frac{x_{3y} + x_{4z}}{2}\sqrt{-p_1^2}\right) J_0\left(\frac{x_{3y} - x_{4z}}{2}\sqrt{-p_2^2}\right) \theta\left(-p_1 p_2 - 2\,\frac{p_1^2 p_2^2}{p_1^2 + p_2^2}\right). \end{aligned}\right\} \tag{A. 19}$$

The two vectors $p_1$ and $p_2$ are both timelike with their sum in the future light cone. The scalar product of two timelike vectors $p_1$ and $p_2$ is either smaller than $-\sqrt{p_1^2 p_2^2}$ when $p_1$ and $p_2$ lie in the same light cone or bigger than $\sqrt{p_1^2 p_2^2}$ when they lie in opposite light cones. The last $\theta$-function in (A. 19) tells us that

$$-p_1 p_2 > 2\,\frac{p_1^2 p_2^2}{p_1^2 + p_2^2} > \sqrt{p_1^2 p_2^2}. \tag{A. 20}$$

Therefore, $p_1$ and $p_2$ both lie in the same light cone. As the sum of $p_1$ and $p_2$ lies in the future light cone, it follows that both vectors $p_1$ and $p_2$ lie in the future light cone. We now write

$$\left.\begin{aligned} \boldsymbol{I}_1 = \iint\limits_{(x,t)} dp_1\, dp_2\, \theta(-p_1^2)\, \theta(-p_2^2)\, \theta(p_1)\, \theta(p_2)\, e^{-ip_{10}\frac{x_{10}+x_{2r}}{2} - ip_{20}\frac{x_{10}-x_{2r}}{2}} \\ \times \left(\frac{p_1^2 - p_2^2}{4}\right)^8 \frac{\theta(A)}{\sqrt{A}}\, J_0\left(\frac{x_{3y} + x_{4z}}{2}\sqrt{-p_1^2}\right) J_0\left(\frac{x_{3y} - x_{4z}}{2}\sqrt{-p_2^2}\right), \end{aligned}\right\} \tag{A. 21}$$

$$A = -1 - \frac{p_1^2 + p_2^2}{2} - \frac{1}{16}(p_1^2 - p_2^2)^2. \tag{A. 21a}$$

If we here introduce "polar coordinates" according to

$$\left.\begin{aligned} p_{ko} &= r_k \cosh \theta_k, \\ p_{kx} &= r_k \sinh \theta_k; \quad k = 1, 2, \end{aligned}\right\} \tag{A. 22}$$

we can perform the integrations over the "angles" $\theta_k$ and obtain

$$\left.\begin{aligned} \boldsymbol{I}_1 = -\pi^2 \iint_0^\infty r_1 r_2\, dr_1\, dr_2 \left(\frac{r_1^2 - r_2^2}{4}\right)^8 \frac{\theta(A)}{\sqrt{A}} H_0^{(1)}\left(-\frac{x_{10} + x_{2x}}{2} r_1\right) H_0^{(1)}\left(-\frac{x_{10} - x_{2x}}{2} r_2\right) \\ \times J_0\left(\frac{x_{3y} + x_{4z}}{2} r_1\right) J_0\left(\frac{x_{3y} - x_{4z}}{2} r_2\right). \end{aligned}\right\} \quad (\text{A. }23)$$

As the second term in (A. 15) can be obtained from the first one if we replace $x_{4z}$ by $-x_{4z}$, we have the following result for the integral $I$ in (33):

$$\left.\begin{aligned} I = \frac{1}{2} \iint_0^\infty dr_1\, dr_2\, F(r_1, r_2)\, \mathrm{H}_0^{(1)}\left(-\frac{x_{10} + x_{2x}}{2} r_1\right) H_0^{(1)}\left(-\frac{x_{10} - x_{2x}}{2} r_2\right) \\ \times \left\{ J_0\left(\frac{x_{3y} + x_{4z}}{2} r_1\right) J_0\left(\frac{x_{3y} - x_{4z}}{2} r_2\right) + J_0\left(\frac{x_{3y} - x_{4z}}{2} r_1\right) J_0\left(\frac{x_{3y} + x_{4z}}{2} r_2\right)\right\}, \end{aligned}\right\} \quad (\text{A. }24)$$

with

$$F(r_1, r_2) = -\pi^4 r_1 r_2 \left(\frac{r_1^2 - r_2^2}{4}\right)^8 \frac{\theta(A)}{\sqrt{A}}, \quad (\text{A. }24\,\text{a})$$

$$A = -1 + \frac{1}{2}(r_1^2 + r_2^2) - \frac{1}{16}(r_1^2 - r_2^2)^2. \quad (\text{A. }24\,\text{b})$$

Eqs. (A. 24) are identical with Eq. (36a) in the main text.

## Appendix III

Consider the following square root:

$$\sqrt{P_1(t)} = \sqrt{(t^2 - t_1^2)(t^2 - t_2^2)}, \quad (\text{A. }25)$$

where $t_1$ and $t_2$ are two fixed complex numbers, both in the upper half-plane, and $t$ is a complex variable. To give this root a well-defined meaning, we introduce two cuts in the complex $t$-plane, one between $t_1$ and $t_2$ and the other between $-t_1$ and $-t_2$, according to Fig. 2. We further require that the square root approaches $t^2$ for large values of $|t|$. The imaginary and real parts of this root change their signs on the cuts and on the curves given by

$$(t^2 - t_1^2)(t^2 - t_2^2) = \varrho, \quad (\text{A. }26)$$

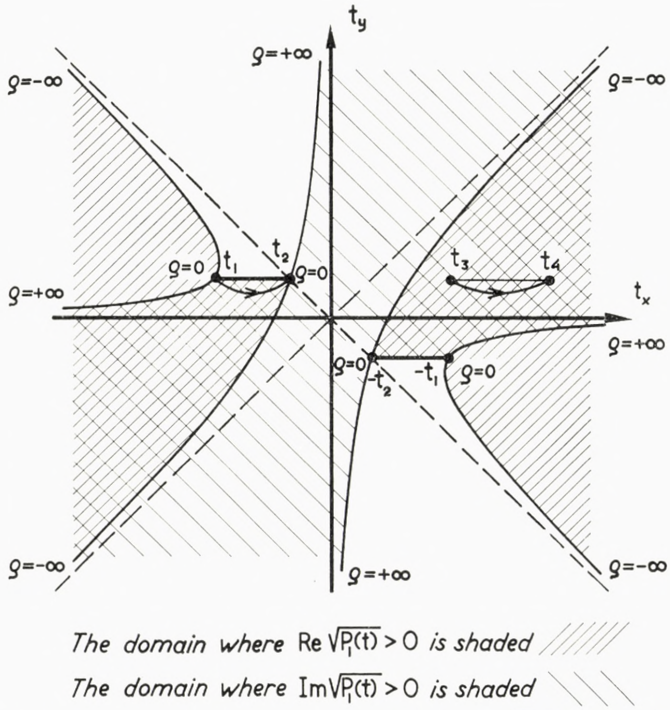


Fig. 2. The cuts in the $t$-plane used to define $\sqrt{P_1(t)}$ in (A. 25).

where $\varrho$ is a real number. When $\varrho$ is positive, the imaginary part of $\sqrt{P_1(t)}$ changes its sign, while the real part changes its sign for negative values of $\varrho$. The curves (A. 26) pass through the points $\pm t_1$ and $\pm t_2$ for $\varrho = 0$. Further, they approach the real and imaginary axes asymptotically for large positive values of $\varrho$. For large negative values of $\varrho$, they instead approach the two lines through the origin with the directions $1 \pm i$. The general behaviour of these curves is indicated in Fig. 2. In the same figure, we have also introduced different shadings for the domains where the real and imaginary parts of $\sqrt{P_1(t)}$ are positive.

In the second term in (41) in the main text, $\sqrt{P_1(t)}$ is used along a path of integration between the points $t_3$ and $t_4$ (cf. Figs. 1 and 2) and is defined to have a positive imaginary part there. According to Fig. 2, this coincides with the definition of $\sqrt{P_1(t)}$ given here.

The first term in (41) contains $\sqrt{-P_1(t)}$ on a path of integration between $t_1$ and $t_2$ and with the definition Im $\sqrt{-P_1(t)} > 0$. According to Fig. 2, Re $\sqrt{P_1(t)}$ is positive on this path, and we conclude

$$\sqrt{-P_1(t)} = +i\sqrt{P_1(t)}. \tag{A. 27}$$

This is one of the relations used in getting from Eq. (41) to Eq. (42) in the main text.

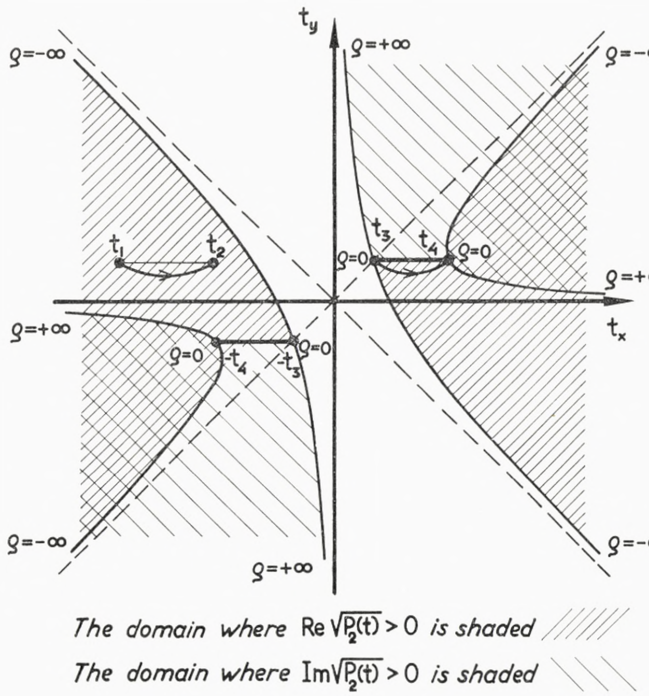


Fig. 3. The cuts in the $t$-plane used to define $\sqrt{P_2(t)}$.

A similar discussion for the polynomial $P_2(t) = (t^2 - t_3^2)(t^2 - t_4^2)$ leads to the cuts and curves shown in Fig. 3. From this diagram we find the relation

$$\sqrt{-P_2(t)} = +i\sqrt{P_2(t)} \tag{A. 28}$$

along the path of integration between $t_3$ and $t_4$, as well as $\sqrt{P_2(t)} \to -\sqrt{P_2(t)}$ on the path between $t_1$ and $t_2$. Eq. (A. 28) is the other relation needed to obtain Eq. (42) in the main text.

*NORDITA – Nordisk Institut for Teoretisk Atomfysik,*
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